The μ- Test

Let $f(x)$ be unbounded at a and integrable in the arbitrary interval $[a+\epsilon, b]$ where $0<\epsilon<b-a$.

If there is a number μ between zero and 1 such that $\lim_{x\to a+0} (x-a)^{\mu} f(x)$ exists, then $\int_a^b f(x)$ converges absolutely.

If there is a number $\mu>1$ or $\mu=1$ s.t $\lim_{x\to a+0} (x-a)^{\mu} f(x)$ exists and is not zero then $\int_a^b f(x)$ diverges and the same is true if $\lim_{x\to a+0} (x-a)^{\mu} f(x) = \pm\infty$.

[Note - Above μ- Test is useful to prove the convergence of Γ function (Gamma) function & Beta function.

Convergence of Gramma Function

prove that $\int_0^\infty x^{n-1} e^{-x} dx$ is Convergent

when $n>0$.

Solⁿ (2) Let $n \geq 1$

Here $f(x) = x^{n-1} e^{-x}$

we write the given integral as

$$\int_0^\infty e^{-x} x^{n-1} dx = \int_0^a e^{-x} x^{n-1} dx + \int_a^\infty e^{-x} x^{n-1} dx \, dx$$

proper Integral + Improper integral

$$I = I_1 + I_2$$

Since I_1 is proper integral.

$\therefore$ it is Cgt.

Now we test the Convergency of I_2

Here $I_2 = \int_0^\infty e^{-x} x^{n-1} dx$

we test the Convergency of I_2 by μ- Test.

We have

$$\lim_{x\to\infty} x^{\mu} f(x) = \lim_{x\to\infty} \frac{x^{\mu} x^{n-1}}{e^{x}}$$

$$= \lim_{x\to\infty} \frac{x^{\mu+n-1}}{e^{x}}$$

$$= \lim_{x\to\infty} \frac{\lfloor \mu+n-1}{e^{x}} = 0 \quad \forall\ \mu \ \& \ n$$

Taking $\mu > 1$, we see that

$$\int_a^{\infty} e^{-x} x^{n-1} dx \text{ is Convergent for } \forall n$$

$$\therefore I = I_1 + I_2$$

$$= Cgt + Cgt = Cgt.$$

$\therefore$ when $n \geq 1$ $\int_0^{\infty} x^{n-1} e^{-x} dx$ is Convergent.

(ii) Let $0 < n < 1$

In this Case $e^{-x} x^{n-1}$ has infinity at $x=0$

Hence we have

$$\int_0^{\infty} e^{-x} x^{n-1} dx = \int_0^{a} e^{-x} x^{n-1} dx + \int_a^{\infty} e^{-x} x^{n-1} dx$$

Now $\lim\limits_{x\to 0} x^{\mu} e^{-x} x^{n-1} = \lim\limits_{x\to 0} e^{-x} x^{\mu+n-1}$

$= \lim\limits_{x\to 0} \dfrac{x^{\mu+n-1}}{e^{x}}$

$= 1$, if $\mu+n-1=0$

i.e $\mu = 1-n$

Since n lies between 0 and 1
therefore μ also lies between 0 and 1

Hence $\int_0^a e^{-x} x^{n-1}\,dx$ is Convergent by μ-test.

Also from (i) $\int_a^{\infty} e^{-x} x^{n-1}\,dx$ is Convergent $\forall n$.

Therefore the given integral is Convergent
when $0<n<1$.

(ii) Let $n \le 0$

In this Case
$f(x) = e^{-x} x^{n-1}$ has infinity
at $x=0$

$\therefore \lim\limits_{x\to 0} x^{\mu} f(x) = \lim\limits_{x\to 0} x^{\mu} \cdot e^{-x} x^{n-1}$

$= \lim\limits_{x\to 0} e^{-x} x^{\mu+n-1}$

$= \infty$

$\mu+n-1<0$
$\mu<1-n$
$\cancel{\mu<1-n}$

$$\therefore \int_0^a e^{-x} x^{n-1} dx \text{ is divergent.}$$

Therefore the given integral is also divergent when $n \leq 0$.

Convergence of Beta Function :-

Discuss the Convergence of the Beta function

$$\int_0^1 x^{m-1} (1-x)^{n-1} dx .$$

Solution (i) when m and n both are ≥ 1, the integrand is finite $\forall\, x$ from 0 to 1.

Hence it is Convergent.

(ii) When m and n are < 1 then the integrand has infinities both at $x=0$ and $x=1$. In this case we write

$$\int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^c x^{m-1}(1-x)^{n-1}dx + \int_c^1 x^{m-1}(1-x)^{n-1}dx$$

$0<c<1$

First we Consider $\int_0^c x^{m-1}(1-x)^{n-1}dx$

Let $f(x) = x^{m-1}(1-x)^{n-1}$ Then

$$\lim_{x\to 0} x^{\mu} f(x) = \lim_{x\to 0} x^{\mu}\cdot x^{m-1}\cdot(1-x)^{n-1}$$

$$= \lim_{x\to 0} x^{\mu+m-1}\cdot(1-x)^{n-1}$$

$= 1$ if $\mu+m-1=0$ i.e if $\mu = 1-m$

Now if $0<m<1$, we have $0<\mu<1$

and if $m \leq 0$, $\mu \geq 1$

Hence by μ test $\int_0^c f(x)dx$ is Convergent if $0<m<1$ and divergent if $m \leq 0$.

Similarly it Can be proved that $\int^1 f(x)dx$ is Convergent if $0<n<1$ and divergent if $n\leq 0$.

Hence from (i) and (ii) it follows that $\int_0^1 f(x)dx$ is Convergent if both m & n are >0 and divergent otherwise.